

UNITED STATES NAVAL POSTGRADUATE SCHOOL



A DETAILED INTEGRATION FOR THE RADIATION IMPEDANCE OF A RHOMBIC ANTENNA

---BY---

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REPORT NUMBER 2

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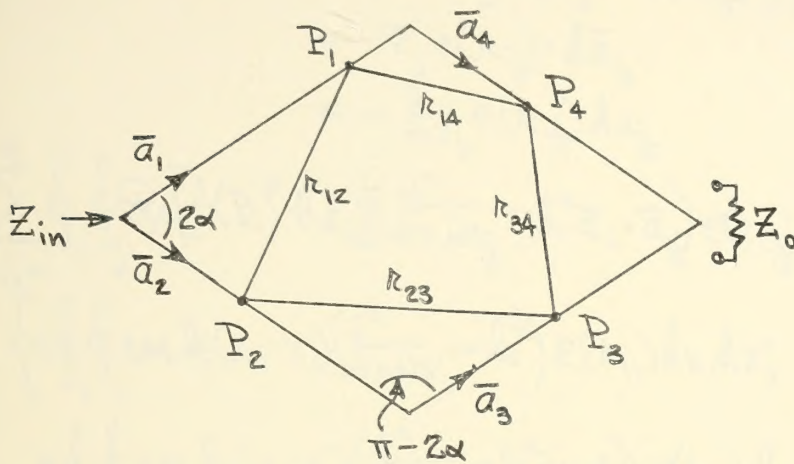
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ABSTRACT

The generalized circuit equation derived by Chaney ⁽¹⁾ is applied directly to evaluation of the radiation impedance of a terminated rhombic antenna in free space. The fact that certain integrations totalize to zero, as demonstrated by Chaney ⁽²⁾ from his equations of constraint, is verified by actual detailed evaluation of the integrals involved. Furthermore, Chaney's formulae ⁽²⁾ for the radiation impedances of both the terminated rhombic and vee in free space are verified by independent integration.

1. J. G. Chaney, "A critical study of the circuit concept", J. Appl. Phys. 22, 12, 1429 (1951).
2. J. G. Chaney, "Free space radiation impedance of rhombic antenna", U. S. Naval Postgraduate School Technical Report No. 4 (BuShips Project Report No. 1) (May 1952).

RADIATION IMPEDANCE OF THE TERMINATED RHOMBIC IN FREE SPACE



$$\begin{aligned} \bar{a}_1 = \bar{a}_3 & \quad \bar{a}_1 \cdot \bar{a}_1 = \bar{a}_2 \cdot \bar{a}_2 = 1 \\ \bar{a}_2 = \bar{a}_4 & \quad \bar{a}_1 \cdot \bar{a}_2 = \cos 2\alpha \end{aligned}$$

$$\bar{I}_1 = \bar{a}_1 I_0 e^{-jkx_1}$$

$$\bar{I}_3 = -\bar{a}_1 I_0 e^{-jk(l+x_3)}$$

$$\bar{I}_2 = -\bar{a}_2 I_0 e^{-jkx_2}$$

$$\bar{I}_4 = \bar{a}_2 I_0 e^{-jk(l+x_4)}$$

where l = length of any leg.

Chaney has derived the following formula for the input impedance of a closed circuit⁽¹⁾,

$$Z_{in} = Z_0 |f_0|^2 + 4l Z_0 f_0^2 + j \frac{30}{k} \underbrace{\oint_a \oint_b \operatorname{Re} [f(P_a)^* f(P_b)] \Delta [e(k_{ab}) d\bar{r}_b] \cdot d\bar{r}_a}_{\stackrel{D}{=} Z_R = \text{radiation impedance}}$$

Thus,

$$Z_R = j \frac{30}{k} \sum_{i,j=1}^4 \oint_i \oint_j \operatorname{Re} [f(P_i)^* f(P_j)] \Delta [e(k_{ij}) d\bar{r}_j] \cdot d\bar{r}_i$$

where $\Delta \stackrel{D}{=} \nabla(\nabla \cdot) + k^2$, $e(k_{ij}) \stackrel{D}{=} \frac{e^{-jk_{ij}r_{ij}}}{r_{ij}}$, $f(P_i) \stackrel{D}{=} I_i$.

$$\begin{aligned}\square_i [e(r_{ij}) d\bar{r}_j] &= \nabla_i \left[\underbrace{\nabla_i \cdot e(r_{ij}) d\bar{r}_j}_{\substack{\downarrow \\ = \nabla_i e(r_{ij}) \cdot d\bar{r}_j + e(r_{ij}) \underbrace{\nabla_i \cdot d\bar{r}_j}_{=0}} \right] + k^2 e(r_{ij}) d\bar{r}_j \\ &= -\nabla_j e(r_{ij}) \cdot d\bar{r}_j \\ &= -\frac{\partial}{\partial x_j} e(r_{ij}) dx_j\end{aligned}$$

$$\begin{aligned}Z_R &= \frac{30}{jk} \sum_{i,j=1}^4 \oint_i \oint_j \oplus_{\mathbf{r}_i} [f(\mathbf{r}_i)^* f(\mathbf{r}_j)] \left(\frac{\partial^2}{\partial x_i \partial x_j} - k^2 \bar{\mathbf{a}}_i \cdot \bar{\mathbf{a}}_j \right) e(r_{ij}) dx_i dx_j \\ &= \frac{30}{jk} \left\{ 4 \oint_1 \oint_1 \cos k(x_1 - x_1') \left(\frac{\partial^2}{\partial x_1 \partial x_1'} - k^2 \right) e(r_{11}) dx_1 dx_1' \right. \\ &\quad - 2 \oint_1 \oint_2 \cos k(x_1 - x_2) \left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2 \cos 2\alpha \right) e(r_{12}) dx_1 dx_2 \\ &\quad - 2 \oint_3 \oint_4 \cos k(x_3 - x_4) \left(\frac{\partial^2}{\partial x_3 \partial x_4} - k^2 \cos 2\alpha \right) e(r_{34}) dx_3 dx_4 \\ &\quad - 2 \oint_1 \oint_3 \cos k(x_1 - x_3 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_3} - k^2 \right) e(r_{13}) dx_1 dx_3 \\ &\quad - 2 \oint_2 \oint_4 \cos k(x_2 - x_4 - l) \left(\frac{\partial^2}{\partial x_2 \partial x_4} - k^2 \right) e(r_{24}) dx_2 dx_4 \\ &\quad + 2 \oint_1 \oint_4 \cos k(x_1 - x_4 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_4} - k^2 \cos 2\alpha \right) e(r_{14}) dx_1 dx_4 \\ &\quad \left. + 2 \oint_2 \oint_3 \cos k(x_2 - x_3 - l) \left(\frac{\partial^2}{\partial x_2 \partial x_3} - k^2 \cos 2\alpha \right) e(r_{23}) dx_2 dx_3 \right\}\end{aligned}$$

$$\oint_1 \oint_2 = \oint_3 \oint_4, \quad \oint_1 \oint_3 = \oint_2 \oint_4, \quad \oint_1 \oint_4 = \oint_2 \oint_3$$

$$r_{11} = \sqrt{a^2 + (x_1 - x_1')^2}$$

$$r_{12} = \sqrt{a^2 + x_1^2 + x_2^2 - 2x_1 x_2 \cos 2\alpha}$$

$$r_{34} = \sqrt{a^2 + (l - x_3)^2 + (l - x_4)^2 - 2(l - x_3)(l - x_4) \cos 2\alpha}$$

$$r_{13} = \sqrt{\alpha^2 + (x_3 - x_1 + l \cos 2\alpha)^2 + (l \sin 2\alpha)^2}$$

$$r_{24} = \sqrt{\alpha^2 + (x_4 - x_2 + l \cos 2\alpha)^2 + (l \sin 2\alpha)^2}$$

$$r_{14} = \sqrt{\alpha^2 + (l - x_1)^2 + x_4^2 + 2(l - x_1)x_4 \cos 2\alpha}$$

$$r_{23} = \sqrt{\alpha^2 + (l - x_2)^2 + x_3^2 + 2(l - x_2)x_3 \cos 2\alpha}$$

Define:

$$Z_{11} = \frac{30}{jk} \int_0^l \int_0^l \cos k(x_1 - x_1') \left(\frac{\partial^2}{\partial x_1 \partial x_1'} - k^2 \right) e(r_{11}) dx_1 dx_1'$$

$$Z_{12} = \frac{30}{jk} \int_0^l \int_0^l \cos k(x_1 - x_2) \left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2 \cos 2\alpha \right) e(r_{12}) dx_1 dx_2$$

$$Z_{13} = \frac{30}{jk} \int_0^l \int_0^l \cos k(x_1 - x_3 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_3} - k^2 \right) e(r_{13}) dx_1 dx_3$$

$$Z_{14} = \frac{30}{jk} \int_0^l \int_0^l \cos k(x_1 - x_4 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_4} - k^2 \cos 2\alpha \right) e(r_{14}) dx_1 dx_4$$

Then

$$Z_R = 4(Z_{11} - Z_{12} - Z_{13} + Z_{14})$$

Subst. $\cos 2\alpha = 1 - 2\sin^2 \alpha$

$$Z_{12} = \frac{30}{jk} \left[\int_0^l \int_0^l \cos k(x_1 - x_2) \left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2 \right) e(r_{12}) dx_1 dx_2 \right. \\ \left. + 2k^2 \sin^2 \alpha \int_0^l \int_0^l \cos k(x_1 - x_2) e(r_{12}) dx_1 dx_2 \right]$$

Z_{14} = "similar expression"

$$\text{Let } A_{ij} = \int_0^l \int_0^l F_{ij}(x_i, x_j) \left(\frac{\partial^2}{\partial x_i \partial x_j} - k^2 \right) e(r_{ij}) dx_i dx_j$$

$$B_{ij} = \int_0^l \int_0^l F_{ij}(x_i, x_j) e(r_{ij}) dx_i dx_j$$

Then

$$Z_R = \frac{120}{jk} [A_{11} - A_{12} - A_{13} + A_{14} - 2k^2 \sin^2 \alpha B_{12} + 2k^2 \sin^2 \alpha B_{14}]$$

$$A_{11} = \int_0^l \int_0^l \cos k(x_1 - x'_1) \left(\frac{\partial^2}{\partial x_1 \partial x'_1} - k^2 \right) e(k_{11}) dx_1 dx'_1$$

$$\text{where } e(k_{11}) = \frac{e^{-i k r_{11}}}{r_{11}}, \quad r_{11} = \sqrt{a^2 + (x_1 - x'_1)^2}$$

$$A_{11} = \int_0^l \int_0^l \cos k(x_1 - x'_1) \frac{\partial^2}{\partial x_1 \partial x'_1} e(k_{11}) dx_1 dx'_1 - k^2 \int_0^l \int_0^l \cos k(x_1 - x'_1) e(k_{11}) dx_1 dx'_1$$

$$= A'_{11} + A''_{11}$$

Integrating by parts

$$\begin{aligned} A'_{11} &= \int_0^l \left[\cos k(x_1 - x'_1) \frac{\partial}{\partial x'_1} e(k_{11}) \right]_0^l dx'_1 + k \int_0^l \int_0^l \sin k(x_1 - x'_1) \frac{\partial}{\partial x'_1} e(k_{11}) dx_1 dx'_1 \\ &= \int_0^l \cos k(l - x'_1) \frac{\partial}{\partial x'_1} e(k_{11}) dx'_1 - \int_0^l \cos k x'_1 \frac{\partial}{\partial x'_1} e(k_{11}) dx'_1 \\ &\quad + k \int_0^l \left[\sin k(x_1 - x'_1) e(k_{11}) \right]_0^l dx_1 + k^2 \int_0^l \int_0^l \cos k(x_1 - x'_1) e(k_{11}) dx_1 dx'_1 \\ &= \cos k(l - x'_1) e(k_{11}) \Big|_0^l - k \int_0^l \sin k(l - x'_1) e(k_{11}) dx'_1 \\ &\quad - \cos k x'_1 e(k_{11}) \Big|_0^l - k \int_0^l \sin k x'_1 e(k_{11}) dx'_1 \\ &\quad + k \int_0^l \sin k(x_1 - l) e(k_{11}) dx_1 - k \int_0^l \sin k x_1 e(k_{11}) dx_1 \\ &\quad + k^2 \int_0^l \int_0^l \cos k(x_1 - x'_1) e(k_{11}) dx_1 dx'_1 \end{aligned}$$

Combining

$$\begin{aligned} A_{11} &= A'_{11} + A''_{11} = 2e(k_{11}) - 2 \cos kl e(k_{11}) - 2k \int_0^l \sin kx_1 e(k_{11}) dx_1 \\ &\quad - 2k \underbrace{\int_0^l \sin k(l - x_1) e(k_{11}) dx_1}_{= \int_0^l \sin kx_1 e(k_{11}) dx_1} \\ &= \int_0^l \sin kx_1 e(k_{11}) dx_1 \end{aligned}$$

$$A_{11} = 2e(k_2 l) - 2 \cos k l e(k_{10}) - 4k \int_0^l \sin k x_1 e(k_{10}) dx_1$$

Evaluate integral in last term

$$\begin{aligned} \int_0^l \sin k x_1 e(k_{10}) dx_1 &= \int_0^l \frac{\sin k x_1 \cos k k_{10}}{k_{10}} dx_1 - j \int_0^l \frac{\sin k x_1 \sin k k_{10}}{k_{10}} dx_1 \\ &= \frac{1}{2} \int_0^l \frac{\sin k(k_{10} + x_1)}{k_{10}} dx_1 + \frac{1}{2} \int_0^l \frac{\sin k(k_{10} - x_1)}{k_{10}} dx_1 \\ &\quad + \frac{j}{2} \int_0^l \frac{\cos k(k_{10} + x_1)}{k_{10}} dx_1 - \frac{j}{2} \int_0^l \frac{\cos k(k_{10} - x_1)}{k_{10}} dx_1 \end{aligned}$$

The following well known integration formulae may be readily verified by the transformation $u = k \pm z$ (+ or - as required).

$$\begin{aligned} \int_{z_1}^{z_2} \frac{\cos \beta(k+z)}{k} dz &= Ci \beta(k_2 + z_2) - Ci \beta(k_1 + z_1) \\ \int_{z_1}^{z_2} \frac{\sin \beta(k+z)}{k} dz &= Si \beta(k_2 + z_2) - Si \beta(k_1 + z_1) \\ \int_{z_1}^{z_2} \frac{\cos \beta(k-z)}{k} dz &= Ci \beta(k_1 - z_1) - Ci \beta(k_2 - z_2) \\ \int_{z_1}^{z_2} \frac{\sin \beta(k-z)}{k} dz &= Si \beta(k_1 - z_1) - Si \beta(k_2 - z_2) \end{aligned} \quad \left\{ \begin{array}{l} \text{where} \\ k = \sqrt{z^2 + \rho^2} \\ \rho = \text{const.} \\ \text{and} \\ Si z \equiv \int_0^z \frac{\sin y}{y} dy \\ Ci z \equiv \int_{-\infty}^z \frac{\cos y}{y} dy \end{array} \right.$$

$$\begin{aligned} \therefore \int_0^l \sin k x_1 e(k_{10}) dx_1 &= \frac{1}{2} \left\{ \underbrace{Si k(k_{10} + l)}_{= Si 2kl} - Si ka - \cancel{Si k(k_{10} - l)} + \cancel{Si ka} \right. \\ &\quad \left. + j \left[\underbrace{Ci k(k_{10} + l)}_{= Ci 2kl} - \underbrace{Ci ka}_{= C + \ln ka} + \underbrace{Ci k(k_{10} - l)}_{= C + \ln k(\sqrt{a^2 + l^2})} - \underbrace{Ci ka}_{= C + \ln k(\frac{a^2}{2l})} \right] \right\} \\ &= \frac{1}{2} \left\{ Si 2kl + j [Ci 2kl - C - \ln 2kl] \right\} \quad \approx C = 0.5772 \end{aligned}$$

Thus

$$A_{11} = \left[\frac{z}{a} - \frac{1 + \cos 2kl}{l} - 2k \operatorname{Si} 2kl \right] \\ + j \left[-2k + \frac{\sin 2kl}{l} - 2k \operatorname{Ci} 2kl + 2k C + 2k \ln 2kl \right]$$

$$A_{12} = \int_0^l \int_0^l \cos k(x_1 - x_2) \left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2 \right) e(k_{12}) dx_1 dx_2$$

$$k_{12} = \sqrt{a^2 + x_1^2 + x_2^2 - 2x_1 x_2 \cos 2\alpha}$$

Integrating by parts as before

$$A_{12} = e(k_{ll}) + e(k_{00}) - 2 \cos kl e(k_{l0}) - 2k \int_0^l \sin kx_1 e(k_{10}) dx_1 \\ - 2k \int_0^l \sin k(l-x_1) e(k_{1l}) dx_1 \quad \begin{cases} k_{ll} = 2l \sin \alpha \\ k_{00} = a \\ k_{l0} = l \end{cases}$$

$$\int_0^l \sin kx_1 e(k_{10}) dx_1 = \frac{1}{2} \int_0^l \frac{\sin 2kx_1}{x_1} dx_1 - j \frac{1}{2} \int_0^l \frac{1 - \cos 2kx_1}{x_1} dx_1 \\ = \frac{1}{2} \left\{ \operatorname{Si} 2kl - j \left[C + \ln 2kl - \operatorname{Ci} 2kl \right] \right\}$$

$$k_{1l} = \sqrt{a^2 + x_1^2 + l^2 - 2x_1 l \cos 2\alpha} = \sqrt{a^2 + (x_1 - l \cos 2\alpha)^2 + l^2 \sin^2 2\alpha}$$

$$\int_0^l \sin k(l-x_1) e(k_{1l}) dx_1 = \int_0^l \frac{\sin k(l-x_1) \cos k k_{1l}}{k_{1l}} dx_1 - j \int_0^l \frac{\sin k(l-x_1) \sin k k_{1l}}{k_{1l}} dx_1$$

Let $x = x_1 - l \cos 2\alpha$, $k_x = \sqrt{a^2 + x^2 + l^2 \sin^2 2\alpha}$.

$$\int_0^l \sin k(l-x_1) e(k_{1l}) dx_1 \\ = \int_{-l \cos 2\alpha}^{l - l \cos 2\alpha} \frac{\sin k(l - l \cos 2\alpha - x) \cos k k_x}{k_x} dx - j \int_{-l \cos 2\alpha}^{l - l \cos 2\alpha} \frac{\sin k(l - l \cos 2\alpha - x) \sin k k_x}{k_x} dx$$

$$\begin{aligned}
&= \sin(2kl\sin^2\alpha) \int_{x_0}^{x_l} \frac{\cos kx \cos kr_x}{k_x} dx - \cos(2kl\sin^2\alpha) \int_{x_0}^{x_l} \frac{\sin kx \cos kr_x}{k_x} dx \\
&\quad - j \sin(2kl\sin^2\alpha) \int_{x_0}^{x_l} \frac{\cos kx \sin kr_x}{k_x} dx + j \cos(2kl\sin^2\alpha) \int_{x_0}^{x_l} \frac{\sin kx \sin kr_x}{k_x} dx \\
&= \frac{1}{2} \sin(2kl\sin^2\alpha) \left[\text{Ci}(k(r_l+x_l)) - \text{Ci}(k(r_0+x_0)) + \text{Ci}(k(r_0-x_0)) - \text{Ci}(k(r_l-x_l)) \right] \\
&\quad - \frac{1}{2} \cos(2kl\sin^2\alpha) \left[\text{Si}(k(r_l+x_l)) - \text{Si}(k(r_0+x_0)) - \text{Si}(k(r_0-x_0)) + \text{Si}(k(r_l-x_l)) \right] \\
&\quad - \frac{j}{2} \sin(2kl\sin^2\alpha) \left[\text{Si}(k(r_l+x_l)) - \text{Si}(k(r_0+x_0)) + \text{Si}(k(r_0-x_0)) - \text{Si}(k(r_l-x_l)) \right] \\
&\quad + \frac{j}{2} \cos(2kl\sin^2\alpha) \left[\text{Ci}(k(r_0-x_0)) - \text{Ci}(k(r_l-x_l)) - \text{Ci}(k(r_l+x_l)) + \text{Ci}(k(r_0+x_0)) \right]
\end{aligned}$$

$$\begin{cases}
r_0+x_0 = \sqrt{l^2 \cos^2 2\alpha + l^2 \sin^2 2\alpha} - l \cos 2\alpha = 2l \sin^2 \alpha \\
r_l+x_l = \sqrt{l^2 (1 - \cos 2\alpha)^2 + l^2 \sin^2 2\alpha} + l \sin 2\alpha = 2l (\sin \alpha + \sin^2 \alpha) \\
r_0-x_0 = 2l \cos^2 \alpha \\
r_l-x_l = 2l (\sin \alpha - \sin^2 \alpha)
\end{cases}$$

Collecting terms-

$$\begin{aligned}
A_{12} = & \left\{ \frac{\cos(2kl\sin^2\alpha)}{2l\sin\alpha} + \frac{1}{a} - 2 \frac{\cos^2 kl}{l} - k \text{Si}(2kl) \right. \\
& - k \sin(2kl\sin^2\alpha) \left[\text{Ci}(2kl\sin\alpha(1+\sin\alpha)) - \text{Ci}(2kl\sin^2\alpha) \right. \\
& \quad \left. + \text{Ci}(2kl\cos^2\alpha) - \text{Ci}(2kl\sin\alpha(1-\sin\alpha)) \right] \\
& + k \cos(2kl\sin^2\alpha) \left[\text{Si}(2kl\sin\alpha(1+\sin\alpha)) - \text{Si}(2kl\sin^2\alpha) \right. \\
& \quad \left. - \text{Si}(2kl\cos^2\alpha) + \text{Si}(2kl\sin\alpha(1-\sin\alpha)) \right] \left. \right\} \\
& + j \left\{ - \frac{\sin(2kl\sin^2\alpha)}{2l\sin\alpha} - k + 2 \frac{\sin kl \cos kl}{l} + k \text{Ci} \right. \\
& \quad \left. + k \ln 2kl - k \text{Ci } 2kl \right\}
\end{aligned}$$

$$\begin{aligned}
& + k \sin(2kl \sin^2 \alpha) \left[\text{Si}(2kl \sin \alpha (1 + \sin \alpha)) - \text{Si}(2kl \sin^2 \alpha) \right. \\
& \quad \left. + \text{Si}(2kl \cos^2 \alpha) - \text{Si}(2kl \sin \alpha (1 - \sin \alpha)) \right] \\
& - k \cos(2kl \sin^2 \alpha) \left[\text{Ci}(2kl \cos^2 \alpha) - \text{Ci}(2kl \sin \alpha (1 - \sin \alpha)) \right. \\
& \quad \left. + \text{Ci}(2kl \sin^2 \alpha) - \text{Ci}(2kl \sin \alpha (1 + \sin \alpha)) \right] \}
\end{aligned}$$

Note: When $\alpha = 0$, reconsideration of the limits of integration readily shows that A_{12} reduces to A_{11} as it should. This serves as a partial check on these integrations.

$$A_{13} = \int_0^l \int_0^l \cos k(x_1 - x_3 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_3} - k^2 \right) e(k_{13}) dx_1 dx_3$$

$$k_{13} = \sqrt{\alpha^2 + (x_3 - x_1 + l \cos 2\alpha)^2 + (l \sin 2\alpha)^2}$$

$$\int_0^l \int_0^l \cos k(x_1 - x_3 - l) \frac{\partial^2}{\partial x_1 \partial x_3} e(k_{13}) dx_1 dx_3$$

$$= \int_0^l \left[\cos k(x_1 - x_3 - l) \frac{\partial}{\partial x_3} e(k_{13}) \right]_0^l dx_3 + k \int_0^l \int_0^l \sin k(x_1 - x_3 - l) \frac{\partial}{\partial x_3} e(k_{13}) dx_1 dx_3$$

$$= \int_0^l \cos kx_3 \frac{\partial}{\partial x_3} e(k_{13}) dx_3 - \int_0^l \cos k(x_3 + l) \frac{\partial}{\partial x_3} e(k_{03}) dx_3$$

$$+ k \int_0^l \left[\sin k(x_1 - x_3 - l) e(k_{13}) \right]_0^l dx_1 + k^2 \int_0^l \int_0^l \cos k(x_1 - x_3 - l) e(k_{13}) dx_1 dx_3$$

$$= \cos kx_3 e(k_{13}) \Big|_0^l + k \int_0^l \sin kx_3 e(k_{13}) dx_3 - \cos k(x_3 + l) e(k_{03}) \Big|_0^l$$

$$- k \int_0^l \sin k(x_3 + l) e(k_{03}) dx_3 + k \int_0^l \sin k(x_1 - 2l) e(k_{1l}) dx_1$$

$$- k \int_0^l \sin k(x_1 - l) e(k_{10}) dx_1 + k^2 \int_0^l \int_0^l \cos k(x_1 - x_3 - l) e(k_{13}) dx_1 dx_3$$

$$A_{13} = \cos kl e(k_{1l}) - e(k_{l0}) - \cos 2kl e(k_{0l}) + \cos kl e(k_{00}) \\ - 2kl \int_0^l \sin k(x_3+l) e(k_{03}) dx_3 - 2kl \int_0^l \sin k(x_1-l) e(k_{10}) dx_1$$

$$\begin{cases} k_{1l} = l \\ k_{l0} = l\sqrt{(1-\cos 2\alpha)^2 + \sin^2 2\alpha} = 2l \sin \alpha \\ k_{0l} = 2l \cos \alpha \\ k_{00} = l \end{cases}$$

$$\begin{aligned} & 2 \int_0^l \sin k(x_3+l) e(k_{03}) dx_3 \quad \begin{cases} k_{03} = \sqrt{(x_3+l \cos 2\alpha)^2 + (l \sin 2\alpha)^2} \\ \text{Let } x = x_3 + l \cos 2\alpha \\ k_x = \sqrt{x^2 + (l \sin 2\alpha)^2} \end{cases} \\ &= 2 \int \frac{l+l \cos 2\alpha}{l \cos 2\alpha} \frac{\sin k(x+l-l \cos 2\alpha) \cos k k_x}{k_x} dx - j 2 \int_{x_0}^{x_l} \frac{\sin k(x+l-l \cos 2\alpha) \sin k k_x}{k_x} dx \\ &= 2 \left\{ \sin(2kl \sin^2 \alpha) \int_{x_0}^{x_l} \frac{\cos k x \cos k k_x}{k_x} dx + \cos(2kl \sin^2 \alpha) \int_{x_0}^{x_l} \frac{\sin k x \cos k k_x}{k_x} dx \right\} \\ &\quad - j 2 \left\{ \sin(2kl \sin^2 \alpha) \int_{x_0}^{x_l} \frac{\cos k x \sin k k_x}{k_x} dx + \cos(2kl \sin^2 \alpha) \int_{x_0}^{x_l} \frac{\sin k x \sin k k_x}{k_x} dx \right\} \\ &= \sin(2kl \sin^2 \alpha) [Cik(k_l+x_l) - Cik(k_0+x_0) + Cik(k_0-x_0) - Cik(k_l-x_l)] \\ &\quad + \cos(2kl \sin^2 \alpha) [Sik(k_l+x_l) - Sik(k_0+x_0) - Sik(k_0-x_0) + Sik(k_l-x_l)] \\ &\quad - j \sin(2kl \sin^2 \alpha) [Sik(k_l+x_l) - Sik(k_0+x_0) + Sik(k_0-x_0) - Sik(k_l-x_l)] \\ &\quad - j \cos(2kl \sin^2 \alpha) [Cik(k_0-x_0) - Cik(k_l-x_l) - Cik(k_l+x_l) + Cik(k_0+x_0)] \\ &\begin{cases} k_0+x_0 = 2l \cos^2 \alpha \\ k_l+x_l = 2l \cos \alpha (1+\cos \alpha) \\ k_0-x_0 = 2l \sin^2 \alpha \\ k_l-x_l = 2l \cos \alpha (1-\cos \alpha) \end{cases} \end{aligned}$$

$$2 \int_0^l \sin k(x-l) e(k_{10}) dx,$$

$$\left\{ \begin{aligned} r_{10} &= \sqrt{(x_1 - l \cos 2\alpha)^2 + (l \sin 2\alpha)^2} \\ \text{let } x &= x_1 - l \cos 2\alpha \\ r_x &= \sqrt{x^2 + (l \sin 2\alpha)^2} \end{aligned} \right.$$

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$$= 2 \int_{-l \cos 2\alpha}^{l - l \cos 2\alpha} \frac{\sin k(x - 2l \sin^2 \alpha) \cos k r_x}{r_x} dx - j 2 \int_{x_0}^{x_1} \frac{\sin k(x - 2l \sin^2 \alpha) \sin k r_x}{r_x} dx$$

= the negative of an integral evaluated for A_{12} above, i.e.,

$$= - \int_0^l \sin k(l - x_1) e(k_{10}) dx_1$$

Collecting results -

$$A_{13} = \left\{ \frac{1 + \cos 2kl}{l} - \frac{\cos(2kl \sin \alpha)}{2l \sin \alpha} - \frac{\cos 2kl \cos(2kl \cos \alpha)}{2l \cos \alpha} \right.$$

$$- kl \sin(2kl \sin^2 \alpha) \left[Ci(2kl \cos \alpha (1 + \cos \alpha)) - 2Ci(2kl \cos^2 \alpha) + 2Ci(2kl \sin^2 \alpha) \right. \\ \left. - Ci(2kl \cos \alpha (1 - \cos \alpha)) - Ci(2kl \sin \alpha (1 + \sin \alpha)) + Ci(2kl \sin \alpha (1 - \sin \alpha)) \right]$$

$$- kl \cos(2kl \sin^2 \alpha) \left[Si(2kl \cos \alpha (1 + \cos \alpha)) - 2Si(2kl \cos^2 \alpha) - 2Si(2kl \sin^2 \alpha) \right. \\ \left. + Si(2kl \cos \alpha (1 - \cos \alpha)) + Si(2kl \sin \alpha (1 + \sin \alpha)) + Si(2kl \sin \alpha (1 - \sin \alpha)) \right] \left. \right\}$$

$$+ j \left\{ - \frac{\sin 2kl}{l} + \frac{\sin(2kl \sin \alpha)}{2l \sin \alpha} + \frac{\cos 2kl \sin(2kl \cos \alpha)}{2l \cos \alpha} \right.$$

$$+ kl \sin(2kl \sin^2 \alpha) \left[Si(2kl \cos \alpha (1 + \cos \alpha)) - 2Si(2kl \cos^2 \alpha) + 2Si(2kl \sin^2 \alpha) \right. \\ \left. - Si(2kl \cos \alpha (1 - \cos \alpha)) - Si(2kl \sin \alpha (1 + \sin \alpha)) + Si(2kl \sin \alpha (1 - \sin \alpha)) \right]$$

$$+ kl \cos(2kl \sin^2 \alpha) \left[2Ci(2kl \sin^2 \alpha) - Ci(2kl \cos \alpha (1 - \cos \alpha)) - Ci(2kl \cos \alpha (1 + \cos \alpha)) \right. \\ \left. + 2Ci(2kl \cos^2 \alpha) - Ci(2kl \sin \alpha (1 - \sin \alpha)) - Ci(2kl \sin \alpha (1 + \sin \alpha)) \right] \left. \right\}$$

// //

$$A_{14} = \int_0^l \int_0^l \cos k(x_1 - x_4 - l) \left(\frac{\partial^2}{\partial x_1 \partial x_4} - k^2 \right) e(k_{14}) dx_1 dx_4$$

$$k_{14} = \sqrt{a^2 + (l - x_1)^2 + x_4^2 + 2(l - x_1)x_4 \cos 2\alpha}$$

Integrating by parts as in A_{13} -

$$A_{14} = \cos k x_4 e(k_{l4}) \Big|_0^l + k \int_0^l \sin k x_4 e(k_{l4}) dx_4 - \cos k(x_4 + l) e(k_{04}) \Big|_0^l$$

$$- k \int_0^l \sin k(x_4 + l) e(k_{04}) dx_4 + k \int_0^l \sin k(x_1 - 2l) e(k_{1l}) dx_1 - k \int_0^l \sin k(x_1 - l) e(k_{10}) dx_1$$

$$\begin{cases} k_{l4} = l \\ k_{l0} = a \end{cases} \quad \begin{cases} k_{0l} = 2l \cos \alpha \\ k_{00} = l \end{cases}$$

$$A_{14} = \cos kl e(k_{l4}) - e(k_{l0}) - \cos 2kl e(k_{0l}) + \cos kl e(k_{00})$$

$$+ 2k \int_0^l \sin k x_4 e(k_{l4}) dx_4 - 2k \int_0^l \sin k(x_4 + l) e(k_{04}) dx_4$$

$$2 \int_0^l \sin k x_4 e(k_{l4}) dx_4 \quad \left\{ \begin{array}{l} k_{l4} = x_4 \end{array} \right.$$

(as in A_{12})

$$= \text{Si } 2kl - i \left[C + \ln 2kl - \text{Ci } 2kl \right]$$

$$2 \int_0^l \sin k(x_4 + l) e(k_{04}) dx_4 \quad \left\{ \begin{array}{l} k_{04} = \sqrt{l^2 + x_4^2 + 2lx_4 \cos 2\alpha} \\ = \sqrt{(x_4 + l \cos 2\alpha)^2 + (l \sin 2\alpha)^2} \\ \text{Let } x = x_4 + l \cos 2\alpha \\ k_x = \sqrt{x^2 + (l \sin 2\alpha)^2} \end{array} \right.$$

(as in A_{13})

$$= \sin(2kl \sin^2 \alpha) \left[\text{Ci}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Ci}(2kl \cos^2 \alpha) + \text{Ci}(2kl \sin^2 \alpha) \right.$$

$$\left. - \text{Ci}(2kl \cos \alpha (1 - \cos \alpha)) \right]$$

$$+ \cos(2kl \sin^2 \alpha) \left[\text{Si}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Si}(2kl \cos^2 \alpha) - \text{Si}(2kl \sin^2 \alpha) \right.$$

$$\left. + \text{Si}(2kl \cos \alpha (1 - \cos \alpha)) \right]$$

$$\begin{aligned}
 & -j \sin(2kl \sin^2 \alpha) \left[\text{Si}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Si}(2kl \cos^2 \alpha) + \text{Si}(2kl \sin^2 \alpha) \right. \\
 & \quad \left. - \text{Si}(2kl \cos \alpha (1 - \cos \alpha)) \right] \\
 & -j \cos(2kl \sin^2 \alpha) \left[\text{Ci}(2kl \sin^2 \alpha) - \text{Ci}(2kl \cos \alpha (1 - \cos \alpha)) - \text{Ci}(2kl \cos \alpha (1 + \cos \alpha)) \right. \\
 & \quad \left. + \text{Ci}(2kl \cos^2 \alpha) \right]
 \end{aligned}$$

Collecting results—

$$\begin{aligned}
 A_{14} = & \left\{ \frac{1 + \cos 2kl}{l} - \frac{1}{a} - \frac{\cos 2kl \cos(2kl \cos \alpha)}{2l \cos \alpha} + k \text{Si } 2kl \right. \\
 & - k \sin(2kl \sin^2 \alpha) \left[\text{Ci}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Ci}(2kl \cos^2 \alpha) \right. \\
 & \quad \left. + \text{Ci}(2kl \sin^2 \alpha) - \text{Ci}(2kl \cos \alpha (1 - \cos \alpha)) \right] \\
 & - k \cos(2kl \sin^2 \alpha) \left[\text{Si}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Si}(2kl \cos^2 \alpha) \right. \\
 & \quad \left. - \text{Si}(2kl \sin^2 \alpha) + \text{Si}(2kl \cos \alpha (1 - \cos \alpha)) \right] \Big\} \\
 & + j \left\{ -\frac{\sin 2kl}{l} + k + \frac{\cos 2kl \sin(2kl \cos \alpha)}{2l \cos \alpha} - k(-k \ln 2kl + k \text{Ci } 2kl \right. \\
 & + k \sin(2kl \sin^2 \alpha) \left[\text{Si}(2kl \cos \alpha (1 + \cos \alpha)) - \text{Si}(2kl \cos^2 \alpha) \right. \\
 & \quad \left. + \text{Si}(2kl \sin^2 \alpha) - \text{Si}(2kl \cos \alpha (1 - \cos \alpha)) \right] \\
 & + k \cos(2kl \sin^2 \alpha) \left[\text{Ci}(2kl \sin^2 \alpha) - \text{Ci}(2kl \cos \alpha (1 - \cos \alpha)) \right. \\
 & \quad \left. - \text{Ci}(2kl \cos \alpha (1 + \cos \alpha)) + \text{Ci}(2kl \cos^2 \alpha) \right] \Big\}
 \end{aligned}$$

Adding up the above integrations with appropriate signs gives

$$A_{11} - A_{12} - A_{13} + A_{14} = 0,$$

hence the radiation impedance may be computed simply from the following expression,

$$Z_r = j 240 k \sin^2 \alpha (B_{12} - B_{14}),$$

which corresponds to Equation (22) of Chaney⁽²⁾.

B_{12} and B_{14} are evaluated below by the method described by Murray⁽³⁾, and for comparison Murray's notation has been essentially retained. He has furthermore shown that, unless the two linear integration paths lie in the same plane, the impedance integrals will lead to untabulated functions. Therefore, B_{12} and B_{14} have been evaluated by taking all paths in the same plane.

$$\begin{aligned}
 B_{12} &= \int_0^l \int_0^l \cos k(x_1 - x_2) e(k_{12}) dx_1 dx_2 \quad \left\{ \begin{array}{l} k_{12} = \sqrt{x_1^2 + x_2^2 - 2x_1 x_2 \cos 2\alpha} \\ \\ \end{array} \right. \\
 &= \underbrace{\frac{1}{2} \int_0^l \int_0^l \frac{e^{jk(x_1 - x_2)}}{k_{12}} e^{-jk_{12}} dx_1 dx_2}_{\equiv G_1} + \underbrace{\frac{1}{2} \int_0^l \int_0^l \frac{e^{-jk(x_1 - x_2)}}{k_{12}} e^{-jk_{12}} dx_1 dx_2}_{\equiv G'_1} \\
 &= \frac{1}{2}(G_1 + G'_1)
 \end{aligned}$$

Interchanging x_1 and x_2 gives $G_1 = G'_1$, hence

$$B_{12} = G_1$$

$$B_{14} = \int_0^l \int_0^l \cos k(x_1 - x_4 - l) e(k_{14}) dx_1 dx_4 \quad \left\{ \begin{array}{l} k_{14} = \sqrt{(l - x_1)^2 + x_4^2 + 2(l - x_1)x_4 \cos 2\alpha} \\ \\ \end{array} \right.$$

Letting $x'_1 = l - x_1$, gives

$$B_{14} = \int_0^l \int_0^l \cos k(x_1 + x_4) e(k_{14}) dx_1 dx_4 \quad \left\{ \begin{array}{l} k_{14} = \sqrt{x_1^2 + x_4^2 + 2x_1 x_4 \cos 2\alpha} \\ \\ \end{array} \right.$$

3. F.H. Murray, "Mutual impedance of two skew antenna wires",
 R. IRE Trans. AP-1, 151 (1953)

$$B_{14} = \frac{1}{2} \underbrace{\int_0^l \int_0^l \frac{e^{jk(x_1+x_4)} e^{-jk r_{14}}}{r_{14}} dx_1 dx_4}_{\substack{\mathcal{D} \\ \equiv G_2}} + \frac{1}{2} \underbrace{\int_0^l \int_0^l \frac{e^{-jk(x_1+x_4)} e^{-jk r_{14}}}{r_{14}} dx_1 dx_4}_{\substack{\mathcal{D} \\ \equiv G_3}}$$

$$= \frac{1}{2} (G_2 + G_3)$$

Now the radiation impedance may be expressed

$$Z_R = j 120k \sin^2 \alpha (2G_1 - G_2 - G_3)$$

// ----- //

Evaluation of G_1 :

$$G_1 = \int_0^l \int_0^l \frac{e^{jk(x_1-x_2-r_{12})}}{r_{12}} dx_1 dx_2$$

$$\begin{aligned} r_{12} &= \sqrt{x_1^2 + x_2^2 - 2x_1 x_2 \cos 2\alpha} \\ &= \sqrt{d_1^2 + (x_2 - cx_1)^2} \\ &= \sqrt{d_2^2 + (x_1 - cx_2)^2} \end{aligned} \quad \left\{ \begin{array}{l} \text{where} \\ d_1 \equiv \sqrt{x_1^2 (1-c^2)} = x_1 s \\ d_2 \equiv \sqrt{x_2^2 (1-c^2)} = x_2 s \\ \text{and } \begin{cases} c \equiv \cos 2\alpha \\ s \equiv \sin 2\alpha \end{cases} \end{array} \right.$$

$$\text{Let } \begin{cases} d_1/t = r_{12} + (x_2 - cx_1) \\ d_1/t = r_{12} - (x_2 - cx_1) \end{cases}$$

$$\text{Then } r_{12} = \frac{d_1}{2} \left(t + \frac{1}{t} \right)$$

$$x_2 - cx_1 = \frac{d_1}{2} \left(t - \frac{1}{t} \right)$$

$$\left. \frac{\partial x_2}{\partial t} \right|_{x_1} = \frac{r_{12}}{t}$$

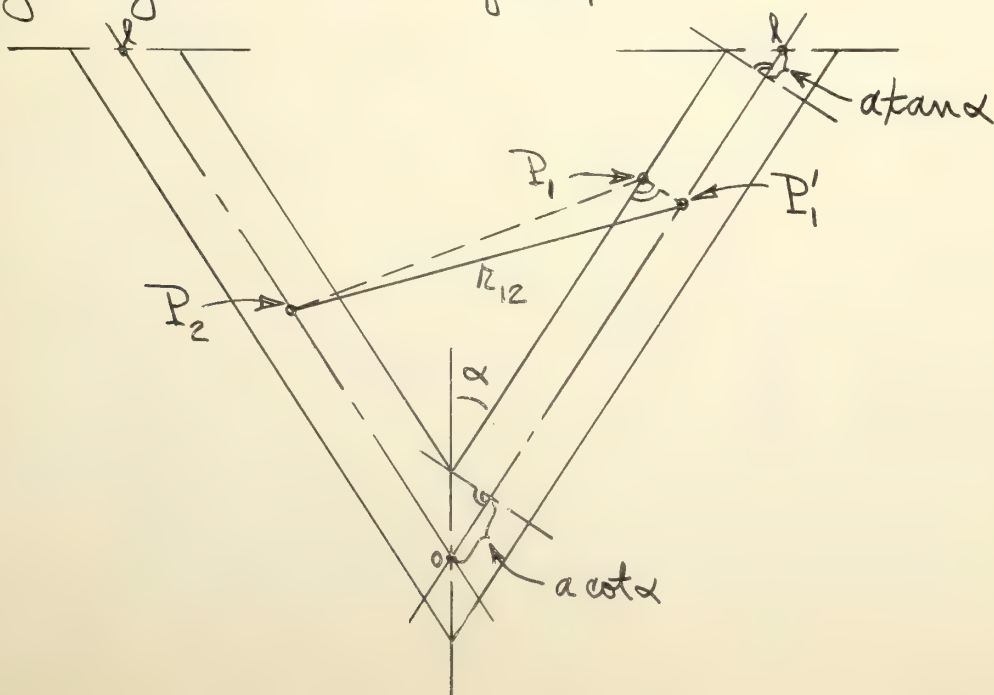
$$G_1 = \int_0^l e^{hx_1} dx_1 \int_0^l \frac{e^{-h(x_1+r_{12})}}{r_{12}} dx_2 \quad \left\{ \begin{array}{l} h \equiv jk \end{array} \right.$$

$$\begin{aligned}
 &= \int_0^l e^{h(1-c)x_1} dx_1 \int_{t_0}^{t_x} \frac{e^{-hdt}}{t} dt = \int_0^l e^{h(1-c)x_1} dx_1 \int_{kx_1(1-c)}^{k[r_{12}+(l-cx_1)]} \frac{e^{-ju}}{u} du \quad 16 \\
 &= \frac{1}{h(1-c)} \left\{ \underbrace{\left[e^{h(1-c)x_1} \int_{2kx_1 \sin^2 \alpha}^{k[r_{12}+(l-cx_1)]} \frac{e^{-ju}}{u} du \right]_0^l}_{\substack{D = F(l) - F(0)}} - \underbrace{\int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hdt}}{dt} \frac{d(dt)}{dx_1} \right]_{dt_0}^{dt_x} dx_1}_{\substack{D = J_1 - J_0}} \right\} \\
 &= \frac{1}{h(1-c)} \left\{ F(l) - F(0) - [J_1 - J_0] \right\}
 \end{aligned}$$

$$[r_{12} + (l - cx_1)]_l = l \sqrt{2(1-c)} + l(1-c) = 2l \sin \alpha (1 + \sin \alpha)$$

$$\begin{aligned}
 F(l) &= e^{j2kl \sin^2 \alpha} \left[Ci(2kl \sin \alpha (1 + \sin \alpha)) - Ci(2kl \sin^2 \alpha) \right. \\
 &\quad \left. - j Si(2kl \sin \alpha (1 + \sin \alpha)) + j Si(2kl \sin^2 \alpha) \right]
 \end{aligned}$$

Selecting integration limits for x_1 :



$$2kx_1 \sin^2 \alpha \Big|_{a \cos \alpha} = ka \sin 2\alpha$$

$$F(0) = Ci(2kl) - C \ln(ka \sin 2\alpha) - j Si(2kl)$$

$$J_x = \int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hd_1 t_1}}{d_1 t_1} \frac{\partial(d_1 t_1)}{\partial x_1} \right] dx_1$$

$$\frac{1}{d_1 t_1} \frac{\partial(d_1 t_1)}{\partial x_1} = \frac{1}{k_{12} + (l - cx_1)} \left(\frac{\partial k_{12}}{\partial x_1} - c \right)$$

Let

$$\begin{cases} dx_1 = k_{12} - (x_1 - cl) \\ dx_1 = k_{12} + (x_1 - cl) \end{cases}$$

Then $k_{12} = \frac{dx_1}{2} \left(u + \frac{1}{u} \right)$

$$x_1 - cl = \frac{dx_1}{2} \left(u - \frac{1}{u} \right)$$

$$\frac{\partial x_1}{\partial u} = -\frac{k_{12}}{u}$$

$$\frac{\partial k_{12}}{\partial x_1} - c = \frac{x_1 - cl - ck_{12}}{k_{12}}$$

$$\begin{aligned} x_1 - cl - ck_{12} &= -\frac{dx_1}{2} \left[u - \frac{1}{u} + c \left(u + \frac{1}{u} \right) \right] = -\frac{dx_1}{2} \left[u(1+c) - \frac{1}{u}(1-c) \right] \\ &= -\frac{dx_1(1+c)}{2u} [u^2 - \gamma^2] \end{aligned}$$

$$\begin{aligned} \gamma^2 &\equiv \frac{1-c}{1+c} \\ \gamma &= \tan \alpha \end{aligned}$$

$$\frac{dx_1}{k_{12}} = -\frac{du}{u}$$

$$k_{12} + (l - cx_1): \quad cx_1 - (1 - s^2)l = -\frac{cdx_1}{2} \left(u - \frac{1}{u} \right)$$

$$k_{12} + l - cx_1 = \frac{dx_1}{2} \left[u + \frac{1}{u} + \frac{2s^2 l}{dx_1} + c \left(u - \frac{1}{u} \right) \right] = \frac{dx_1}{2u} (u + \gamma)^2$$

$$J_x = e^{-h(1-c)l} \int_{u_0}^{u_1} \frac{e^{-hdx_1} (u - \tan \alpha)}{u(u + \tan \alpha)} du$$

$$\begin{cases} dx_1 u_1 = k_{12} - l(1-c) = 2l \sin \alpha (1 - \sin \alpha) \\ dx_1 u_0 = l(1+c) = 2l \cos^2 \alpha \end{cases}$$

$$J_1 = e^{-j2kl\sin^2\alpha} \left\{ -\int_{u_0}^{u_2} \frac{e^{-hdu}}{u} du + 2 \int_{u_0}^{u_2} \frac{e^{-hdu}}{u + \tan\alpha} du \right\}$$

$$= e^{-j2kl\sin^2\alpha} \left\{ -\int_{2kl\cos^2\alpha}^{2kl\sin\alpha(1-\sin\alpha)} \frac{e^{-jw}}{w} dw + 2e^{j2kl\sin^2\alpha} \int_{2kl}^{2kl\sin\alpha} \frac{e^{-jw}}{w} dw \right\}$$

$$\begin{cases} du_2 + d\tan\alpha = 2l\sin\alpha(1-\sin\alpha) + 2l\sin^2\alpha = 2l\sin\alpha \\ du_0 + d\tan\alpha = 2l \end{cases}$$

$$J_1 = -e^{-j2kl\sin^2\alpha} \left[Ci(2kl\sin\alpha(1-\sin\alpha)) - Ci(2kl\cos^2\alpha) \right. \\ \left. - jSi(2kl\sin\alpha(1-\sin\alpha)) + jSi(2kl\cos^2\alpha) \right] \\ + 2 \left[Ci(2kl\sin\alpha) - Ci(2kl) - jSi(2kl\sin\alpha) - jSi(2kl) \right]$$

$$d_1 t_0 = k_{10} - cx_1 = x_1(1-c)$$

$$J_0 = \int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hd_1 t_0}}{d_1 t_0} \frac{\partial(d_1 t_0)}{\partial x_1} \right] dx_1 = \int_0^l e^{h(1-c)x_1} \left[\frac{e^{-h(1-c)x_1}}{(1-c)x_1} (1-c) \right] dx_1$$

$$= \int_{\cot\alpha}^l \frac{dx_1}{x_1} = \ln\left(\frac{l\tan\alpha}{a}\right)$$

Collecting results

$$G_1 = \frac{1}{j2kl\sin^2\alpha} \left\{ \ln(2kl\sin^2\alpha) + 1 + Ci(2kl) - 2Ci(2kl\sin\alpha) \right. \\ + \cos(2kl\sin^2\alpha) [Ci(2kl\sin\alpha(1+\sin\alpha)) - Ci(2kl\sin^2\alpha)] \\ + \sin(2kl\sin^2\alpha) [Si(2kl\sin\alpha(1+\sin\alpha)) - Si(2kl\sin^2\alpha)] \\ + \cos(2kl\sin^2\alpha) [Ci(2kl\sin\alpha(1-\sin\alpha)) - Ci(2kl\cos^2\alpha)] \\ \left. - \sin(2kl\sin^2\alpha) [Si(2kl\sin\alpha(1-\sin\alpha)) - Si(2kl\cos^2\alpha)] \right\}$$

$$\begin{aligned}
& +j2Si(2kl\sin\alpha) - jSi(2kl) \\
& -j\cos(2kl\sin^2\alpha) [Si(2kl\sin\alpha(1+\sin\alpha)) - Si(2kl\sin^2\alpha)] \\
& +j\sin(2kl\sin^2\alpha) [Ci(2kl\sin\alpha(1+\sin\alpha)) - Ci(2kl\sin^2\alpha)] \\
& -j\cos(2kl\sin^2\alpha) [Si(2kl\sin\alpha(1-\sin\alpha)) - Si(2kl\cos^2\alpha)] \\
& -j\sin(2kl\sin^2\alpha) [Ci(2kl\sin\alpha(1-\sin\alpha)) - Ci(2kl\cos^2\alpha)] \}
\end{aligned}$$

$$G_2 = \int_0^l \int_0^l \frac{e^{jk(x_1+x_4-k_{14})}}{r_{14}} dx_1 dx_4$$

$$\begin{aligned}
r_{14} &= \sqrt{x_1^2 + x_4^2 + 2x_1x_4c} \\
&= \sqrt{d_1^2 + (x_4+cx_1)^2} \\
&= \sqrt{d_4^2 + (x_1+cx_4)^2}
\end{aligned}
\left\{ \begin{array}{l} \text{where} \\ d_1 = x_1s \\ d_4 = x_4s \end{array} \right.$$

$$d_1t = r_{14} - (x_4 + cx_1) \quad r_{14} = \frac{d_1}{2} \left(t + \frac{1}{t} \right)$$

$$d_1/t = r_{14} + (x_4 + cx_1) \quad x_4 + cx_1 = -\frac{d_1}{2} \left(t - \frac{1}{t} \right)$$

$$\frac{\partial x_4}{\partial t} = -\frac{r_{14}}{t}$$

$$G_2 = \int_0^l e^{hx_1} dx_1 \int_0^l \frac{e^{h(x_4-k_{14})}}{r_{14}} dx_4 = - \int_0^l e^{h(1-c)x_1} dx_1 \int_{t_0}^{t_l} \frac{e^{-hd_1t}}{t} dt$$

$$= - \int_0^l e^{h(1-c)x_1} dx_1 \int_{\frac{k[k_{14}-(1-cx_1)]}{2kx_1\sin^2\alpha}}^{\frac{k[k_{14}-(1-cx_1)]}{2kx_1\sin^2\alpha}} \frac{e^{-\frac{du}{u}}}{u} du$$

$$= -\frac{1}{h(1-c)} \left\{ \underbrace{\left[e^{h(1-c)x_1} \int_{\frac{k d_1 t_0}{2k d_1 t_0}}^{\frac{k d_1 t_l}{2k d_1 t_0}} \frac{e^{-\frac{du}{u}}}{u} du \right]_0^l}_{F(l) - F(0)} - \underbrace{\int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hd_1t}}{d_1t} \frac{\partial(d_1t)}{\partial x_1} \right]_{d_1t_0}^{d_1t_l} dx_1}_{J_l - J_0} \right\}$$

$$F(l) = e^{jk(1-c)l} \left\{ Ci(2kl \cos \alpha (1 - \cos \alpha)) - Ci(2kl \sin^2 \alpha) \right. \\ \left. - jSi(2kl \cos \alpha (1 - \cos \alpha)) + jSi(2kl \sin^2 \alpha) \right\}$$

$$k_{1l} = 2l \cos \alpha$$

$$k_{1l} - l(1+c) = 2l \cos \alpha (1 - \cos \alpha)$$

$$[k_{1l} - (l + cx_1)]_{\tan \alpha} = \sqrt{l^2 + 2altan \alpha \cos 2\alpha + a^2 \tan^2 \alpha} - (l + a \tan \alpha \cos 2\alpha) \\ = \frac{2a^2}{l} \sin^4 \alpha$$

$$[2x_1 \sin^2 \alpha]_{\tan \alpha} = 2a \frac{\sin^3 \alpha}{\cos \alpha}$$

$$F(0) = Ci\left(\frac{2ka^2}{l} \sin^4 \alpha\right) - Ci\left(2ka \frac{\sin^3 \alpha}{\cos \alpha}\right) = \ln\left(\frac{a}{l} \sin \alpha \cos \alpha\right)$$

$$J_l = \int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hd_1 t_l}}{d_1 t_l} \frac{\partial(d_1 t_l)}{\partial x_1} \right] dx_1$$

$$\frac{1}{d_1 t_l} \frac{\partial(d_1 t_l)}{\partial x_1} = \frac{1}{k_{1l} - (l + cx_1)} \left(\frac{\partial k_{1l}}{\partial x_1} - c \right)$$

$$d_1 u = k_{1l} - (x_1 + cl)$$

$$k_{1l} = \frac{d_1}{2} \left(u + \frac{1}{u} \right)$$

$$d_2 u = k_{1l} + (x_1 + cl)$$

$$x_1 + cl = -\frac{d_2}{2} \left(u - \frac{1}{u} \right)$$

$$\frac{\partial x_1}{\partial u} = -\frac{k_{1l}}{u}$$

$$\frac{\partial k_{1l}}{\partial x_1} - c = \frac{x_1 - cl - ck_{1l}}{k_{1l}}$$

$$x_1 - cl - ck_{1l} = -\frac{d_1}{2} \left[u - \frac{1}{u} + c \left(u + \frac{1}{u} \right) \right]$$

$$= -\frac{d_1(1+c)}{2u} (u^2 - \gamma^2) \quad \gamma^2 = \frac{1-c}{1+c}$$

$$\frac{dx_1}{k_{1l}} = -\frac{du}{u}$$

$$\gamma = \tanh \alpha$$

$$r_{1l} - l - cx_1 =$$

$$cx_1 + (1-s^2)l = -\frac{d_2 c}{2} \left(u - \frac{1}{u}\right)$$

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$$r_{1l} - l - cx_1 = \frac{d_2}{2} \left[u + \frac{1}{u} - \frac{2ls^2}{d_2} + c \left(u - \frac{1}{u}\right) \right]$$

$$= \frac{d_2(1+c)}{2u} \left(u - \frac{1}{u}\right)^2$$

$$J_1 = e^{jk(1-c)l} \int_{u_0}^{u_1} \frac{e^{-hd_2 u} (u + \tan \alpha)}{u(u - \tan \alpha)} du$$

$$d_2 u_1 = r_{1l} - l(1+c) = 2l \cos \alpha (1 - \cos \alpha)$$

$$d_2 u_0 = r_{0l} - cl = 2l \sin^2 \alpha$$

$$J_1 = e^{jk(1-c)l} \left\{ - \int_{u_0}^{u_1} \frac{e^{-hd_2 u}}{u} du + 2 \int_{u_0}^{u_1} \frac{e^{-hd_2 u}}{u - \tan \alpha} du \right\}$$

$$= e^{j2kl \sin^2 \alpha} \left\{ - \int_{kd_2 u_0}^{kd_2 u_1} \frac{e^{-jw}}{w} dw + 2e^{-hd_2 \tan \alpha} \int_{kd_2 (\tan \alpha - u_0)}^{kd_2 (\tan \alpha - u_1)} \frac{e^{+jw}}{w} dw \right\}$$

$$d_2 (\tan \alpha - u_1) = d_2 \tan \alpha - d_2 u_1 = 2l(1 - \cos \alpha)$$

$$d_2 (\tan \alpha - u) = 2l \sin^2 \alpha - \sqrt{l^2 + 2lcx_1 + x_1^2} + (x_1 + cl)$$

$$d_2 (\tan \alpha - u) \Big|_{at \tan \alpha} = \frac{2ka \sin^3 \alpha}{\cos \alpha}$$

$$J_1 = - \underbrace{e^{j2kl \sin^2 \alpha} \int_{2kl \sin^2 \alpha}^{2kl \cos \alpha (1 - \cos \alpha)} \frac{e^{-jw}}{w} dw}_{F(l)} + 2 \int_{\frac{2ka \sin^3 \alpha}{\cos \alpha}}^{2kl(1 - \cos \alpha)} \frac{e^{+jw}}{w} dw$$

$$= -F(l) + 2 \left\{ Ci \langle 2kl(1 - \cos \alpha) \rangle + j Si \langle 2kl(1 - \cos \alpha) \rangle - C - \ln \frac{2ka \sin^3 \alpha}{\cos \alpha} \right\}$$

$$d_1 t_0 = r_{10} - cx_1 = x_1(1-c)$$

$$J_0 = \int_0^l e^{h(1-c)x_1} \left[\frac{e^{-h(1-c)x_1}}{x_1(1-c)} (1-c) \right] dx_1 = \int_{at \tan \alpha}^l \frac{dx_1}{x_1}$$

$$= \ln \left(\frac{l}{at \tan \alpha} \right)$$

$$\begin{aligned}
 G_2 &= -\frac{1}{j2k\sin^2\alpha} \left\{ F(l) - F(0) - J_l + J_0 \right\} \\
 &= -\frac{1}{j2k\sin^2\alpha} \left\{ 2\ln(2kl\sin^2\alpha) + 2C - 2Ci\langle 2kl(1-\cos\alpha) \rangle - 2\ln(\cos\alpha) \right. \\
 &\quad + 2\cos(2kl\sin^2\alpha) \left[Ci\langle 2kl\cos\alpha(1-\cos\alpha) \rangle - Ci(2kl\sin^2\alpha) \right] \\
 &\quad + 2\sin(2kl\sin^2\alpha) \left[Si\langle 2kl\cos\alpha(1-\cos\alpha) \rangle - Si(2kl\sin^2\alpha) \right] \\
 &\quad - j2\cos(2kl\sin^2\alpha) \left[Si\langle 2kl\cos\alpha(1-\cos\alpha) \rangle - Si(2kl\sin^2\alpha) \right] \\
 &\quad + j2\sin(2kl\sin^2\alpha) \left[Ci\langle 2kl\cos\alpha(1-\cos\alpha) \rangle - Ci(2kl\sin^2\alpha) \right] \\
 &\quad \left. - j2Si\langle 2kl(1-\cos\alpha) \rangle \right\}
 \end{aligned}$$

$$G_3 = \int_0^l \int_0^l \frac{e^{-jk(x_1+x_4+k_{14})}}{k_{14}} dx_1 dx_4$$

$$d_1 t = k_{14} + (x_4 + cx_1) \quad k_{14} = \frac{d_1}{2} \left(t + \frac{1}{t} \right)$$

$$d_1/t = k_{14} - (x_4 + cx_1) \quad x_4 + cx_1 = \frac{d_1}{2} \left(t - \frac{1}{t} \right)$$

$$\frac{\partial x_4}{\partial t} = \frac{k_{14}}{t}$$

$$\begin{aligned}
 G_3 &= \int_0^l e^{-h(1-c)x_1} dx_1 \int_{t_0}^{t_l} \frac{e^{-h d_1 t}}{t} dt = \int_0^l e^{-h(1-c)x_1} dx_1 \int_{k d_1 t_0}^{k d_1 t_l} \frac{e^{-j u}}{u} du \\
 &= -\frac{1}{h(1-c)} \left\{ \underbrace{\left[e^{-h(1-c)x_1} \int_{2kx_1 \cos^2\alpha}^{k[k_{14} + (l+cx_1)]} \frac{e^{-j u}}{u} du \right]}_{F(l) - F(0)} - \underbrace{\int_0^l e^{-h(1-c)x_1} \left[\frac{e^{-h d_1 t}}{d_1 t} \frac{\partial(d_1 t)}{\partial x_1} \right]_{d_1 t_0}^{d_1 t_l} dx_1}_{J_l - J_0} \right\}
 \end{aligned}$$

$$k_{14} = 2l \cos\alpha$$

$$k_{14} + l(1+c) = 2l \cos\alpha (1 + \cos\alpha)$$

$$F(l) = e^{-j2kl \sin^2 \alpha} \left\{ Ci(2kl \cos \alpha (1 + \cos \alpha)) - Ci(2kl \cos^2 \alpha) \right. \\ \left. - j Si(2kl \cos \alpha (1 + \cos \alpha)) + j Si(2kl \cos^2 \alpha) \right\}$$

$$[k_{1x} + (l + ex_1)]_0 = 2l$$

$$2kx_1 \cos^2 \alpha \Big|_{\alpha = \alpha_0} = k a \sin 2\alpha$$

$$F(0) = Ci(2kl) - C - \ln(k a \sin 2\alpha) - j Si(2kl)$$

$$J_x = \int_0^l e^{-h(1-c)x_1} \left[\frac{e^{-h d_1 t_x}}{d_1 t_x} \frac{\partial(d_1 t_x)}{\partial x_1} \right] dx_1$$

$$\frac{1}{d_1 t_x} \frac{\partial(d_1 t_x)}{\partial x_1} = \frac{1}{k_{1x} + (l + ex_1)} \left(\frac{\partial k_{1x}}{\partial x_1} + c \right)$$

$$d_x u = k_{1x} + (x_1 + cl) \quad k_{1x} = \frac{d_x}{2} \left(u + \frac{1}{u} \right)$$

$$d_y u = k_{1x} - (x_1 + cl) \quad x_1 + cl = \frac{d_y}{2} \left(u - \frac{1}{u} \right)$$

$$\frac{\partial x_1}{\partial u} = \frac{k_{1x}}{u}$$

$$\frac{\partial k_{1x}}{\partial x_1} + c = \frac{x_1 + cl + c k_{1x}}{k_{1x}}$$

$$x_1 + cl + c k_{1x} = \frac{d_x}{2} \left[u - \frac{1}{u} + c \left(u + \frac{1}{u} \right) \right]$$

$$= \frac{d_x(1+c)}{2u} (u^2 - y^2) \quad \left\{ \begin{array}{l} y^2 = \frac{1-c}{1+c} \\ y = \tan \alpha \end{array} \right.$$

$$\frac{dx_1}{k_{1x}} = \frac{du}{u}$$

$$k_{1x} + l + ex_1: \quad cx_1 + (1-s^2)l = \frac{cd_x}{2} \left(u - \frac{1}{u} \right)$$

$$k_{1x} + l + ex_1 = \frac{d_x}{2} \left[u + \frac{1}{u} + \frac{2s^2 l}{d_x} + c \left(u - \frac{1}{u} \right) \right]$$

$$= \frac{d_x}{2u} (u + y)^2$$

$$\begin{aligned}
 J_1 &= e^{-h(1-c)l} \int_{u_0}^{u_2} \frac{e^{-hdu} (u - \tan \alpha)}{u(u + \tan \alpha)} du \\
 &= e^{-jzkl \sin^2 \alpha} \left\{ - \int_{u_0}^{u_2} \frac{e^{-hdu}}{u} du + z \int_{u_0}^{u_2} \frac{e^{-hdu}}{u + \tan \alpha} du \right\} \\
 &= e^{-jzkl \sin^2 \alpha} \left\{ - \int_{d_2 u_0}^{d_2 u_2} \frac{e^{-jw}}{w} dw + z e^{jzkl \sin^2 \alpha} \int_{d_2(u_0 + \tan \alpha)}^{d_2(u_2 + \tan \alpha)} \frac{e^{-jw}}{w} dw \right\}
 \end{aligned}$$

$$d_2 u_2 = r_{2l} + (1+c)l = 2l \cos \alpha (1 + \cos \alpha)$$

$$d_2 u_0 = r_{0l} + cl = 2l \cos^2 \alpha$$

$$d_2 u_2 + d_2 \tan \alpha = 2l(1 + \cos \alpha)$$

$$d_2 u_0 + d_2 \tan \alpha = 2l$$

$$\begin{aligned}
 J_1 &= -F(l) + z \left[Ci(zkl(1 + \cos \alpha)) - Ci(zkl) \right. \\
 &\quad \left. - j Si(zkl(1 + \cos \alpha)) + j Si(zkl) \right]
 \end{aligned}$$

$$d_1 t_0 = r_{10} + cx_1 = x_1(1+c)$$

$$J_0 = \int_0^1 e^{-h(1-c)x_1} \left[\frac{e^{-h(1+c)x_1}}{x_1(1+c)} (1+c) \right] dx_1 = \int_0^1 \frac{e^{-jzkl x_1}}{x_1} dx_1$$

$$= \int_{zkl \tan \alpha}^{zkl} \frac{e^{-jw}}{w} dw$$

$$= Ci(zkl) - c - \ln(zkl \tan \alpha) - j Si(zkl)$$

$$\begin{aligned}
 G_3 &= - \frac{1}{jzkl \sin^2 \alpha} \left\{ 2Ci(zkl) + 2\ln(\cos \alpha) - 2Ci(zkl(1 + \cos \alpha)) \right. \\
 &\quad \left. + 2\cos(zkl \sin^2 \alpha) [Ci(zkl \cos \alpha (1 + \cos \alpha)) - Ci(zkl \cos^2 \alpha)] \right. \\
 &\quad \left. - 2\sin(zkl \sin^2 \alpha) [Si(zkl \cos \alpha (1 + \cos \alpha)) - Si(zkl \cos^2 \alpha)] \right\}
 \end{aligned}$$

$$\begin{aligned}
& +j[2\text{Si}(2kl(1+\cos\alpha)) - 2\text{Si}(2kl)] \\
& -j2\cos(2kl\sin^2\alpha)[\text{Si}(2kl\cos\alpha(1+\cos\alpha)) - \text{Si}(2kl\cos^2\alpha)] \\
& -j2\sin(2kl\sin^2\alpha)[\text{Ci}(2kl\cos\alpha(1+\cos\alpha)) - \text{Ci}(2kl\cos^2\alpha)] \\
& \hline
\end{aligned}$$

Now from p. 15

$$\frac{Z_R}{120} = jk\sin^2\alpha(2G_1 - G_2 - G_3)$$

Substituting for the G 's now gives

$$\begin{aligned}
\frac{Z_R}{120} = & 2C + 2\ln(2kl\sin^2\alpha) + 2\text{Ci}(2kl) - 2\text{Ci}(2kl\sin\alpha) \\
& - \text{Ci}(2kl(1+\cos\alpha)) - \text{Ci}(2kl(1-\cos\alpha)) \\
& + \cos(2kl\sin^2\alpha)[\text{Ci}(2kl\cos\alpha(1-\cos\alpha)) + \text{Ci}(2kl\sin\alpha(1-\sin\alpha)) \\
& + \text{Ci}(2kl\cos\alpha(1+\cos\alpha)) + \text{Ci}(2kl\sin\alpha(1+\sin\alpha)) \\
& - 2\text{Ci}(2kl\cos^2\alpha) - 2\text{Ci}(2kl\sin^2\alpha)] \\
& + \sin(2kl\sin^2\alpha)[\text{Si}(2kl\cos\alpha(1-\cos\alpha)) - \text{Si}(2kl\sin\alpha(1-\sin\alpha)) \\
& - \text{Si}(2kl\cos\alpha(1+\cos\alpha)) + \text{Si}(2kl\sin\alpha(1+\sin\alpha)) \\
& + 2\text{Si}(2kl\cos^2\alpha) - 2\text{Si}(2kl\sin^2\alpha)] \\
& + j\{\text{Si}(2kl(1+\cos\alpha)) - \text{Si}(2kl(1-\cos\alpha)) + 2\text{Si}(2kl\sin\alpha) \\
& - 2\text{Si}(2kl) \\
& + \cos(2kl\sin^2\alpha)[-\text{Si}(2kl\cos\alpha(1-\cos\alpha)) - \text{Si}(2kl\sin\alpha(1-\sin\alpha)) \\
& - \text{Si}(2kl\cos\alpha(1+\cos\alpha)) - \text{Si}(2kl\sin\alpha(1+\sin\alpha)) \\
& + 2\text{Si}(2kl\cos^2\alpha) + 2\text{Si}(2kl\sin^2\alpha)] \\
& + \sin(2kl\sin^2\alpha)[\text{Ci}(2kl\cos\alpha(1-\cos\alpha)) - \text{Ci}(2kl\sin\alpha(1-\sin\alpha)) \\
& - \text{Ci}(2kl\cos\alpha(1+\cos\alpha)) + \text{Ci}(2kl\sin\alpha(1+\sin\alpha)) \\
& + 2\text{Ci}(2kl\cos^2\alpha) - 2\text{Ci}(2kl\sin^2\alpha)]\}
\end{aligned}$$

RADIATION IMPEDANCE OF THE TERMINATED VEE IN FREE SPACE

The radiation impedance of the vee may be readily computed now from Chaney's formula for the input impedance of the closed circuit* except that now the open end of the vee is assumed closed through the terminating impedance. If the leg length is l and the vertex angle 2α as before, and the same notation is employed as for the rhombic, the obvious result is obtained

$$Z_R = 2(Z_{11} - Z_{12})$$

$$= \frac{60}{jk} (A_{11} - A_{12} - 2k^2 \sin^2 \alpha B_{12})$$

or

$$\frac{Z_R}{60} = \frac{1}{jk} (A_{11} - A_{12}) + j 2k \sin^2 \alpha G_1$$

Substitution gives

$$\frac{Z_R}{60} = \frac{\sin(2kl \sin \alpha)}{2kl \sin \alpha} - 1 + 2 \left[C + \ln(2kl \sin \alpha) - \text{Ci}(2kl \sin \alpha) \right]$$

$$+ j \left[2 \text{Si}(2kl \sin \alpha) + \frac{\cos(2kl \sin \alpha)}{2kl \sin \alpha} - \frac{1}{ka} \right]$$

This expression may be simplified by introducing the modified cosine integral

$$\text{Cin } x = C + \ln x - \text{Ci } x$$

* See p.2.

and by noting that the reactive components

$$\frac{\cos(2kl\sin\alpha)}{2kl\sin\alpha} - \frac{1}{ka}$$

represent the capacitance existing between the open ends of the vee. Since this capacitive reactance is shunted by the terminating impedance Z_0 , these latter terms may be discarded as being negligible. Thus finally

$$\frac{Z_R}{Z_0} = \frac{\sin(2kl\sin\alpha)}{2kl\sin\alpha} - 1 + 2 \cos(2kl\sin\alpha) + j 2 \sin(2kl\sin\alpha)$$

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